

FINAL REPORT

RESEARCH ON PLASMA TURBULENCE
INVOLVING BINARY PARTICLE COLLISIONS
AND
COLLECTIVE EFFECTS

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by
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PREFACE

This report summarizes theoretical analyses carried out under contracts NASW-1778, "Research in the Theory of Magneto-hydrodynamic Turbulence," and NASW-2082, "Theoretical Research on Plasma Turbulence Involving Binary Particle Collisions and Effects," over the period August 1968 to June 1971 and NASW-2352, "Research on Plasma Turbulence Involving Binary Particle Collisions and Collective Effects," over the period February 1972 to May 1972.

The publications that have been made under this contract are:

1. F.X. Murphy and G. Sandri, "Causality of the Nonequilibrium Correlation Functions, Phys. Rev. Lett. 27, 381 (1971).
2. A. Klimas, R.V. Ramnath and G. Sandri, "On the Compatibility Problem for the Uniformization of Asymptotic Expansions," J. Math. Analys, and Applications 32, 482 (1970).
3. A. Klimas, F.X. Murphy and G. Sandri, "Poincare-Lighthill and Linear Time Scales Methods for Linear Time Scales Methods for Linear Perturbation Problems," Quart. Appl. Math., June 1972.
4. F.X. Murphy and G. Sandri, "Causality and Dispersion Relations for the Two-particle Correlation Function," Bull. Am. Phys. Soc. 16, 552 (1971).
5. A. Klimas and G. Sandri, "Asymptotic Expansions in Hilbert Space," Not. Am. Math. Soc. 18, 527 (1971).
6. F.X. Murphy and G. Sandri, "Uniformization of Asymptotic Expansions," Not. Am. Math. Soc. 18, 928 (1971).
7. A.H. Kritz and G. Sandri, "On the Quasi-linear Method in Plasma Turbulence," Bull. Am. Phys. Soc. 15, 96 (1970).

The following three papers are in preparation:

8. A.H. Kritz and G. Sandri, "The Ionization Model in Non-equilibrium Statistical Mechanics."
9. F. Hanson, G.V. Ramanathan, A. Klimas and G. Sandri, "Model for Motion of Charged Particles in the Presence of Electrostatic or Magnetostatic Fluctuations."
10. F.X. Murphy and G. Sandri, "General Dispersion Relations for the Two-particle Correlation Function."

I. OUTLINE OF THE PROBLEM, MAIN RESULTS AND CONCLUSIONS

The main purpose of our study is to investigate plasmas in which binary collisions are important ("energetic" plasmas) as contrasted with "collisionless" plasmas (this latter being of more direct interest to the fusion program). Under a number of conditions which occur frequently, it is necessary to study these plasmas by means of nonadiabatic methods (as contrasted to the adiabatic techniques exemplified by Bogolubov's method). A most important example of such circumstances is:

1. A microunstable particle distribution function.

Other important examples are:

2. The presence of rapidly oscillating external fields (i.e., when the external frequency is comparable to plasma frequency, Oberman and Dawson),
3. The presence of persistent two-body correlations in a stable plasma (Kritz, Ramanathan and Sandri),
4. The presence of moderate external magnetic fields even if not rapidly oscillating (i.e., when the Larmor radius is comparable with the Debye length, Klimas and Sandri).

In outline, we can contrast the adiabatic point of view with the nonadiabatic one as follows. If we don't use the adiabatic approximation, we can formally solve the equation for the two-particle correlation, g , with the help of a Green's function operator, G , as

$$g(t) = \int_0^t G(t-\lambda)F(\lambda)d\lambda \quad (1.1)$$

and then obtain a formal equation for the one-particle distribution, F , as

$$\begin{aligned} \frac{DF}{Dt} &= \int_0^t LG(t-\lambda)F(\lambda)d\lambda \\ &= \int_0^t K(t-\lambda)F(\lambda)d\lambda \end{aligned} \quad (1.2)$$

where L is a linear ("phase-mixing") operator and

$$K(t) \equiv LG(t) \quad (1.3)$$

Approximations to the quantities G and LG have been studied by a number of authors (Lenard, Dupree, Gurnsey, Wolff, C. Wu, Rogister and Oberman). The adiabatic approximation to Eq. (1.1) consists of writing (on the basis that F varies slowly)

$$g_{ad}(t) = \int_0^\infty G(\lambda) d\lambda F_{ad}(t) \quad (1.4)$$

and the corresponding adiabatic approximation to Eq. (1.2) is then

$$\begin{aligned} \left(\frac{DF(t)}{Dt} \right)_{ad} &= L \int_0^\infty G(\lambda) d\lambda F_{ad}(t) \\ &\equiv C[F_{ad}(t)] \end{aligned} \quad (1.5)$$

which is the standard form of the kinetic equation for a stable plasma (the Fokker-Planck equation of Landau or the Bogolubov-Balescu-Lenard-Gurnsey modification of it). When either of the two following quantities contain divergences

$$\int_0^\infty G(\lambda) d\lambda, \quad L \int_0^\infty G(\lambda) d\lambda$$

the adiabatic approximation clearly fails.

When the adiabatic kinetic equation, Eq. (1.5), holds, one can deduce from it a set of moment equations for the density, flow velocity and absolute temperature that allow for a fluid description (provided, of course, that the system is not rarefied). In the cases where nonadiabatic treatment is needed, one can try to solve Eq. (1.2) or, equivalently, one can try to solve simultaneously for the pair of quantities (F, g) . (Some authors replace g by the density of field fluctuations I and then consider the pair (F, I)). Meaningful moment equations for the nonadiabatic system are at present not available in general. (This is also

the case for the fully collisionless regimes. The moment equations of the Chew-Low-Goldberger theory are only semimicroscopic.) An attractive possibility is that under suitable conditions one may succeed in deriving from the nonadiabatic pair (F,I) coupled equations for the particle kinetic energy density and the field energy density which coincide with those proposed by Tchen.

It is impossible, at present, to obtain a general tractable simplification of the nonadiabatic equations. We have, however, been able to obtain general conditions that must be satisfied by the real and imaginary parts of the Fourier transform of g, \tilde{g} . These relations are in the form of standard dispersion relations, e.g.,

$$\text{Re } \tilde{g}(0) = \frac{1}{\pi} P \int \frac{\text{Im } \tilde{g}(\omega')}{\omega - \omega'} d\omega' \quad (1.6)$$

Since the dispersion relations have been shown to be valid without the use of the adiabatic approximation (Eq. (1.6) is a general consequence of the "causality" of particle propagation), they offer a test for any approximation scheme one may try. For example, they are violated in the "ionization model" (i.e., if the potential is switched on at $t = 0$) if the adiabatic approximation is applied. Furthermore, since $\text{Im } \tilde{g}$ determines DF/Dt quite directly and $\text{Re } \tilde{g}$ is, in principle, measurable (by the scattering of weak radiation from a plasma), one may hope that the dispersion relations may one day help to coordinate experimental facts much as the standard dispersion relations for quantum systems have done.

In view of the difficulties in treating the nonadiabatic equations directly, it is important to obtain approximation methods that give useful simplifications. We have obtained two schemes: (1) the "reordering technique" (Ramanathan and Sandri) that has been applied successfully to determine the behavior of the long-range part of the stable plasma g function (the resulting g satisfies the dispersion relations (Eq. (1.6)) and (2) the "compatible" multiple scale expansion (Klimas and Sandri) which has

been successful to construct the evolution of F for the ionization model. Our results seem to agree with the independent analysis of Humphrey, Wilson and Futterman, "Solution of the Rate Equations by Multiple Time Scale Method," Lockheed Lab. preprint (1971).

Since a general approximation scheme to handle the nonadiabatic equations is not available,* we have used numerical methods to investigate their behavior by means of:

- (1) One-dimensional models for charged particles in the presence of electrostatic fluctuations which have nontrivial features in common with two-stream or loss cone microinstabilities (i.e., a Penrose zero in the dielectric function).
- (2) Models for charged particles in the presence of a three-dimensional isotropic spectrum of magnetostatic fluctuations. We find a diffusion tensor substantially different from the adiabatic one.

The numerical method that we have employed successfully consists in Laplace transforming Eq. (1.2) and then inverting its solution with the Bellman-Kalaba-Lockett technique. The numerical integration scheme has proved quite successful even though the systems considered do not resemble those originally discussed by Bellman, Kalaba and Lockett.

The conclusions that arise from our analytic study of the ionization model and from the numerical integrations of systems with static fluctuations is that substantial numerical, as well as qualitative, differences occur between the adiabatic and non-adiabatic approaches. In particular,

1. For one-dimensional electrostatic fluctuations, the kernel function decays as $K(t) \sim 1/t$ giving rise to an F that grows (or decays) substantially more rapidly than in the adiabatic approximation. Furthermore, F exhibits a very long tail

* The situation is again analogous to that holding in the collisionless regime where numerical integration of the Vlasov equation has successfully paralleled the results of the quasi-linear theory for a "bump in the tail" distribution.

($F \sim [t(\ln t)^2]^{-1}$) which is barely integrable. Such a tail is totally absent in the adiabatic treatment. The nonadiabatic total area under F is exactly zero, while in the adiabatic treatment it is very large (or not defined).

2. In the ionization model, the correlations persist for all times in sharp contrast with the adiabatic theory which reaches instant equilibrium. Again, the adiabatic theory does not allow for the long tail in the F function.

3. In the models with magnetostatic fluctuations, the diffusion tensor D_{ad} must be replaced by a diffusion function $D(t)$. The quantity

$$D_{ad} \equiv D(\infty) \quad (1.7)$$

often is infinite or vanishes in sharp contrast with the results of adiabatic approximation.

A general feature that emerges from the nonadiabatic theory is that transport is not local in time. Thus, for example, the heat flow vector is of the form

$$\vec{q}(t) = -\int_0^t h(t-\lambda) \vec{\nabla} T(\lambda) d\lambda \quad (1.8)$$

rather than the total (memoryless) adiabatic result

$$\vec{q}_{ad} = -\left[\int_0^\infty h(\lambda) d\lambda \right] \vec{\nabla} T(t) = -K \vec{\nabla} T(t) \quad (1.9)$$

The numerical analysis shows that the transport coefficients can differ by substantial factors (several decades) from the adiabatic results. We believe that this result is particularly significant because static and steady-state configurations, as well as the stability of a plasma with collisions, depend on the transport properties.

The outline of this report is as follows. In Section II, we give the calculations of the two- and three-body correlations

needed to determine F for the ionization model. (The correlations can be treated adiabatically in this case.) In Section III, we summarize the general dispersion analysis and discuss the examples of the ionization model and of the static fluctuations.

II. ANALYSIS OF THE CORRELATION FUNCTIONS

We consider the Lorentz gas from first principles. It is constituted by N ions and one electron (equivalently a large number of noninteracting electrons). The Hamiltonian is given by

$$H^N = \frac{p^2}{2m} + \sum_{n=1}^N \phi(|\vec{x} - \vec{x}_1|) \quad (2.1)$$

The mass of the electron can be taken to be one and the potential to be monotonic.

We have Newton's equations which are written as

$$\dot{\vec{x}} = \frac{\partial H^N}{\partial \vec{p}} = \vec{p} = \vec{v} \quad (2.2)$$

$$\dot{\vec{p}} = - \frac{\partial H^N}{\partial \vec{x}} = - \sum_{i=1}^N \frac{\partial \phi(|\vec{x} - \vec{x}_1|)}{\partial \vec{x}} = - \sum_{i=1}^N \frac{\partial \phi_1}{\partial \vec{x}} = - \frac{\partial \Phi}{\partial \vec{x}} \quad (2.3)$$

where we have introduced the total potential for the electron

$$\Phi = \sum_{i=1}^N \phi(|\vec{x} - \vec{x}_1|) \quad (2.4)$$

The system is contained in a large box of volume V with specularly reflecting walls. We then have Liouville's theorem which can be written as

$$\begin{aligned} \frac{DF^N}{Dt} &= 0 = \frac{\partial F^N}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial F^N}{\partial \vec{x}} + \dot{\vec{p}} \cdot \frac{\partial F^N}{\partial \vec{p}} \\ &= \frac{\partial F^N}{\partial t} + \vec{v} \cdot \vec{\nabla} F^N - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \end{aligned} \quad (2.5)$$

We introduce the reduced distribution functions by

$$F^s = \int F^N \prod_{i=s+1}^N \frac{dx_i}{V} \quad (2.6)$$

where s is the number of ions. The equation satisfied by the reduced distribution functions is obtained from the Liouville theorem as follows

$$\frac{\partial F^s}{\partial t} = \int \frac{\partial F^N}{\partial t} \prod_{i=s+1}^N \frac{dx_i}{V} = \int \left\{ -\vec{v} \cdot \vec{\nabla}_F^N + \frac{\partial \phi}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \right\} \prod_{i=s+1}^N \frac{dx_i}{V} \quad (2.7)$$

The first term can be transformed as follows

$$\int \vec{v} \cdot \vec{\nabla}_F^N \prod_{i=s+1}^N \frac{dx_i}{V} = \vec{v} \cdot \vec{\nabla} \int F^N \prod_{i=s+1}^N \frac{dx_i}{V} = \vec{v} \cdot \vec{\nabla} F^s \quad (2.8)$$

The second term in Eq. (2.7) can be transformed as follows

$$\begin{aligned} & \int \sum_{i=1}^N \frac{\partial \phi_i}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \prod_{j=s+1}^N \frac{dx_j}{V_j} = \\ & \int \sum_{i=1}^s \frac{\partial \phi_i}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \prod_{j=s+1}^N \frac{dx_j}{V_j} + \int \sum_{i=s+1}^N \frac{\partial \phi_i}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \prod_{j=s+1}^N \frac{dx_j}{V_j} \end{aligned} \quad (2.9)$$

The first part of this quantity can be written simply as

$$I^s F^s \quad (2.10)$$

The second part in Eq. (2.9),

$$\sum_{i=s+1}^N \int \frac{\partial \phi_i}{\partial \vec{x}} \cdot \frac{\partial F^N}{\partial \vec{v}} \prod_{j=s+1}^N \frac{dx_j}{V_j} \quad (2.11)$$

can be reduced considerably. Using the symmetry of the F function, we have $N-s$ equal terms. Therefore, the second contribution to (2.9), given by (2.11), can be written

$$\frac{N-s}{V} \int \frac{\partial \phi_{s+1}}{\partial \vec{x}} \cdot \frac{\partial F^{s+1}}{\partial \vec{v}} d\vec{x}_{s+1} \quad (2.12)$$

Summarizing, we can write the Liouville equation in the following form

$$\frac{\partial F^s}{\partial t} = -\vec{v} \cdot \vec{\nabla} F^s + I^s F^s + \frac{N-s}{V} L_{s+1} F^{s+1} \quad (2.13)$$

with an obvious notation for L . We have carried out the reduction of the distribution function without introducing dimensionless parameters. We now study the dimensions of the relevant quantities.

From Eq. (2.6), we can prove readily

$$F^s = \int F^{s+1} \frac{d\vec{x}_{s+1}}{V} \quad (2.14)$$

Therefore, dimensionally, we have

$$[F^s] = [F^{s+1}] \quad (2.15)$$

Therefore, the only quantities that carry relevant dimensions are the operators.

We introduce dimensionless variables as follows

$$\phi' = \frac{\phi}{\phi_0}, \quad t' = t \frac{v_{th}}{r_0}, \quad \vec{v}' = \frac{\vec{v}}{v_{th}}, \quad \vec{x}' = \frac{\vec{x}}{r_0} \quad (2.16)$$

We now multiply Eq. (2.13) by the quantity

$$\frac{r_0}{v_{th}} \quad (2.17)$$

We then obtain

$$x \frac{r_0}{v_{th}} \frac{\partial F^s}{\partial t} + \frac{r_0}{v_{th}} \vec{v} \cdot \vec{\nabla} F^s = \frac{r_0}{v_{th}} \sum_{i=1}^s \frac{\partial \phi_i}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{v}} F^s + \frac{r_0}{v_{th}} \frac{N-s}{V} \int d^3 x_{s+1} \frac{\partial \phi_{s+1}}{\partial \vec{x}} \cdot \frac{\partial F^{s+1}}{\partial \vec{v}} \quad (2.18)$$

Using the dimensionless variable (2.16), we can rewrite

$$\frac{\partial F^s}{\partial t} + \vec{v}' \cdot \vec{\nabla}' F^s = \frac{\phi_0}{v_{th}^2} \sum \frac{\partial \phi'}{\partial \vec{x}'} \frac{\partial F^s}{\partial \vec{v}'} + \frac{r_0^3 \phi_n}{v_{th}^2} \int d^3 x'_{s+1} \frac{\partial \phi'}{\partial \vec{x}'} \cdot \frac{\partial F^{s+1}}{\partial \vec{v}'} \quad (2.19)$$

We will use the following parameters

$$v_{th}^2 = kT \quad , \quad \epsilon = \frac{\phi_0}{kT} \quad , \quad nr_0^3 \sim 1 \quad (2.20)$$

We can introduce now the Meyer expansion given by

$$F^s = k^0 + \sum_{i=1}^s k'_i + \sum_{i=1}^{s-1} \sum_{j=2}^s k_{ij}^2 + \dots \quad (2.21)$$

This gives the hierarchy for the correlation functions as

$$\frac{\partial k^s}{\partial t} + \vec{v} \cdot \vec{\nabla} k^s = \epsilon I^s k^s + \epsilon \sum_{i=1}^s I_i k_{(i)}^{s-1} + \epsilon \int d\vec{x}_{s+1} I_{s+1} k^{s+1} \quad (2.22)$$

The last term in Eq. (2.22) can be written

$$L_{s+1} k^{s+1} \quad (2.23)$$

Now, we carry out the direct expansion of the hierarchy for the correlation equations. For weakly coupled systems, we can write

$$\frac{\partial k^s}{\partial t} + \vec{v} \cdot \vec{\nabla} k^s = \epsilon I^s k^s + \epsilon \sum_{i=1}^s I_i k_{(i)}^{s-1} + \epsilon L_{s+1} k^{s+1} \quad (2.24)$$

The following ordering is now introduced

$$k^s = \sum_{n=0}^{\infty} \epsilon^n k^{s(n)} \quad (2.25)$$

In zero order, we obtain

$$\frac{\partial k^{s(0)}}{\partial t} + \vec{v} \cdot \vec{\nabla} k^{s(0)} = 0 \quad (2.26)$$

whose solution is given by

$$k^{s(0)}(t) = e^{-\vec{v} \cdot \vec{\nabla} t} k^{s(0)}(0) \quad (2.27)$$

It is convenient to use the following variables

$$\vec{\xi}_1 = \vec{x} - \vec{x}_1 \quad (2.28)$$

We can then write Eq. (2.27) more explicitly as

$$k^s(0)(\vec{\xi}_1, \vec{v}, t) = k^s(0)(\vec{\xi}_1 - \vec{v}t, \vec{v}, 0) \quad (2.29)$$

The use of variable (2.28) is suggested by spatial homogeneity which we shall assume throughout the rest of the calculation.

In first order, we obtain

$$\begin{aligned} \frac{\partial k^s(1)}{\partial t} + \vec{v} \cdot \vec{\nabla} k^s(1) &= I^s e^{-\vec{v} \cdot \vec{\nabla} t} k^s(0)(0) + \sum_{i=1}^s I_i e^{-\vec{v} \cdot \vec{\nabla} t} k_{(i)}^{s-1}(0) + \\ &+ \int d\vec{x}_{s+1} I_{s+1} e^{-\vec{v} \cdot \vec{\nabla} t} k^{s+1}(0) \end{aligned} \quad (2.30)$$

The solution to this equation is given by

$$\begin{aligned} k^s(1)(t) &= e^{-\vec{v} \cdot \vec{\nabla} t} k^s(1)(0) + e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} I^s e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^s(0)(0) + \\ &+ e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} \sum_{i=1}^s I_i e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^{s-1}(0)(0) + \\ &+ e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} \int d\vec{x}_{s+1} I_{s+1} e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^{s+1}(0)(0) \end{aligned} \quad (2.31)$$

In second order, we obtain the following equation

$$\frac{\partial k^s(2)}{\partial t} + \vec{v} \cdot \vec{\nabla} k^s(2) = I^s k^s(1)(t) + \sum_{i=1}^s I_i k_{(i)}^{s-1}(1)(t) + \int d\vec{x}_{s+1} I_{s+1} k^{s+1}(1)(t) \quad (2.32)$$

We now examine the case $s = 0$ which yields

$$\frac{\partial k^{0(2)}}{\partial t} = \int d\vec{x}_1 I_1 k^{1(1)}(t) \quad (2.33)$$

Specializing Eq. (2.31) for the case $s = 1$, we obtain

$$\begin{aligned}
k^1(1) = & e^{-\vec{v} \cdot \vec{\nabla} t} k^1(1)(0) + e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} I_1 e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^1(0)(0) + \\
& + e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} \int d\vec{x}_2 I_2 e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^2(0)(0) \\
& + e^{-\vec{v} \cdot \vec{\nabla} t} \int_0^t e^{\vec{v} \cdot \vec{\nabla} \lambda} I_1 e^{-\vec{v} \cdot \vec{\nabla} \lambda} d\lambda k^0(0)(0) \quad (2.34)
\end{aligned}$$

This expression can be manipulated into the following form

$$\begin{aligned}
k^1(1) = & k^1(1)(\vec{x}-\vec{x}_1-\vec{v}t, \vec{v}, 0) + \int_0^t d\lambda \frac{\partial \phi}{\partial \vec{x}}(\vec{x}-\vec{x}_1-\vec{v}\lambda) \left(\frac{\partial}{\partial \vec{x}} + \lambda \frac{\partial}{\partial \vec{v}} \right) k^1(0)(\vec{x}-\vec{x}_1-\vec{v}t, \vec{v}, 0) + \\
& + \int_0^t d\lambda \int d\vec{x}_2 \frac{\partial \phi}{\partial \vec{x}}(\vec{x}-\vec{x}_2-\vec{v}\lambda) \left(\frac{\partial}{\partial \vec{x}} + \lambda \frac{\partial}{\partial \vec{v}} \right) k^2(0)(\vec{x}-\vec{x}_1-\vec{v}t, \vec{x}-\vec{x}_2-\vec{v}t, \vec{v}, 0) + \\
& + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(\vec{x}-\vec{x}_1-\vec{v}\lambda) d\lambda \frac{\partial k^0(0)}{\partial \vec{v}}(0) \quad (2.35)
\end{aligned}$$

The second term in Eq. (2.35) can be written

$$\begin{aligned}
& \int_0^t e^{-\vec{v} \cdot \vec{\nabla} (t-\lambda)} I_1 e^{-\vec{v} \cdot \vec{\nabla}} e^{+\vec{v} \cdot \vec{\nabla} t} d\lambda e^{-\vec{v} \cdot \vec{\nabla} t} k^1(0)(0) \\
& = \left[\int_0^t d\lambda e^{-\vec{v} \cdot \vec{\nabla} \lambda} I_1 e^{\vec{v} \cdot \vec{\nabla} \lambda} \right] k^1(0)(\vec{x}-\vec{x}_1-\vec{v}t, \vec{v}, 0) \\
& = \int_0^t d\lambda \frac{\partial \phi}{\partial \vec{x}}(\vec{x}-\vec{x}_1-\vec{v}\lambda) \left(\frac{\partial}{\partial \vec{x}} + \frac{\partial}{\partial \vec{v}} \right) k^1(0)(\vec{x}-\vec{x}_1-\vec{v}t, \vec{v}, 0) \quad (2.36)
\end{aligned}$$

The part containing $k^1(0)$ is a transient in time so that this quantity, as well as the third term, cannot contribute any secularity. We notice, however, that the fourth term in Eq. (2.35) contains $k^0(0)$ at time equal zero which is constant in space. Therefore, we can have a secularity. We conclude that $k^1(1)$ has a transient term - one term that saturates to a constant in time. We can write therefore

$$k^{0(2)} = k^{0(2)} + \int_0^t d\lambda \int d\vec{x}_1 I_1 k^{1(1)}(\lambda) \quad (2.37)$$

which, for large values of t , introduces a secularity. We have

$$\begin{aligned} k^{0(2)} &= k^{0(2)}(0) + \int d\vec{x}_1 \frac{\partial \phi(\vec{x} - \vec{x}_1)}{\partial \vec{x}} \int_0^t d\lambda k^{1(1)}(\lambda) \\ &= k^{0(2)}(0) + \int d\vec{x}_1 \frac{\partial \phi(\vec{x} - \vec{x}_1)}{\partial \vec{x}} \int_0^t d\lambda \int_0^\lambda \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} d\mu \frac{\partial k^{0(0)}}{\partial \vec{v}}(0) \end{aligned} \quad (2.38)$$

To exhibit the secularity, we examine carefully the integral

$$\begin{aligned} \int_0^t d\lambda \int_0^\lambda \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} d\mu &= \int_0^t d\mu \int_\mu^t d\lambda \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} \\ &= \int_0^t d\mu \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} [t - \mu] \end{aligned} \quad (2.39)$$

This quantity can be manipulated into the form

$$= t \int_0^t d\mu \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} - \int_0^t \mu d\mu \frac{\partial \phi(\vec{x} - \vec{x}_1 - \vec{v}\mu)}{\partial \vec{x}} \quad (2.40)$$

Both integrals reach a maximum (or saturate) when

$$t \sim \frac{|\vec{x} - \vec{x}_1| + r_0}{v} \quad (2.41)$$

but the coefficient on the first term grows with time. Therefore, it is a true secularity.

We examine the transient term. We notice that if the quantity

$$k^{1(1)}(\vec{x} - \vec{x}_1 - \vec{v}t, \vec{v}, 0) \quad (2.42)$$

is bounded in space, i.e., if

$$|\vec{x} - \vec{x}_1 - \vec{v}\lambda| > R \Rightarrow k = 0 \quad (2.43)$$

then we obtain

$$\int_0^t d\lambda \, k \underset{t \uparrow}{\sim} \text{constant} \quad (2.44)$$

For the other terms, again we can reverse the order of integration, giving, for example,

$$\int_0^t d\mu \, \frac{\partial \mathcal{J}}{\partial \vec{x}} (\vec{x} - \vec{x}_1 - \vec{v}\mu) \left(\frac{\partial}{\partial \vec{v}} + \mu \frac{\partial}{\partial \vec{x}} \right) \int_\mu^t d\lambda k^{(1)}(\vec{x} - \vec{x}_1 - \vec{v}, \vec{v}, 0) \quad (2.45)$$

Here, the second integral saturates to a constant in time. Clearly, we do have a secularity which we must eliminate by uniformizing the expansion.

Therefore, we introduce the extended functions

$$\underline{k}^S(\xi_1, v, \tau_0, \tau_1, \dots) \quad (2.46)$$

which satisfy

$$\underline{k}^S(\xi_1, v, t, \epsilon t, \epsilon^2 t, \dots) = k^S(\xi_1, v, t) \quad (2.47)$$

and we have chosen the trajectory in extended space

$$\tau_n = \epsilon^n t \quad (2.48)$$

so that the time derivatives can be written

$$\frac{D\underline{k}^S}{Dt} = \frac{\partial \underline{k}^S}{\partial \tau_0} + \epsilon \frac{\partial \underline{k}^S}{\partial \tau_1} + \epsilon^2 \frac{\partial \underline{k}^S}{\partial \tau_2} + O(\epsilon^3) \quad (2.49)$$

We now expand the extended functions as follows

$$\underline{k}^s = \underline{k}^{s(0)} + \epsilon \underline{k}^{s(1)} + \epsilon^2 \underline{k}^{s(2)} + o(\epsilon^3) \quad (2.50)$$

where the correction terms should not contain secularities. From this point on, the quantities are expanded extended functions and we drop the bars under the functions.

Now, we consider the zeroth-order equation. We obtain

$$\frac{\partial k^{0(0)}}{\partial \tau_0} = 0 \quad (2.51)$$

The solution of this equation is given by

$$k^{0(0)}(\vec{\tau}) = k^{0(0)}(\vec{\tau}_1) \quad (2.52)$$

For s greater than zero, we obtain

$$\frac{\partial k^{s(0)}}{\partial \tau_0} + \vec{v} \cdot \vec{\nabla} k^{s(0)} = 0 \quad (2.53)$$

whose solution can be written as

$$k^{s(0)} = k^{s(0)}(\xi_1 - v\tau_0, v, 0, \vec{\tau}_1) \quad (2.54)$$

The first-order equation is given by

$$\frac{\partial k^{0(1)}}{\partial \tau_0} + \frac{\partial k^{0(0)}}{\partial \tau_1} = \int d\vec{x}_1 \frac{\partial \phi}{\partial \vec{x}}(\vec{x} - \vec{x}_1) \frac{\partial k^{1(0)}}{\partial \vec{v}}(\vec{x} - \vec{x}_1 - \vec{v}\tau_0, \vec{v}, 0, \vec{\tau}_1) \quad (2.55)$$

We examine $s = 0$ first and find

$$\begin{aligned} k^{0(1)} &= k^{0(1)}(0, \vec{\tau}_1) + \int_0^\tau d\lambda \int d\vec{x}_1 \frac{\partial \phi}{\partial \vec{x}}(x - x_1) \frac{\partial k^{1(0)}}{\partial \vec{v}}(x - x_1 - v\lambda, v, 0, \vec{\tau}_1) + \\ &- \tau_0 \frac{\partial k^{0(0)}}{\partial \tau_1}(\vec{\tau}_1) \end{aligned} \quad (2.56)$$

We eliminate the second term which saturates because

$$\frac{\partial k^{0(0)}}{\partial \tau_1} = 0 \quad (2.57)$$

so that we obtain

$$k^{0(0)}(\vec{\tau}) = k^{0(0)}(\vec{\tau}_2) \quad (2.58)$$

This equation replaces Eq. (2.52). Therefore, we are left with

$$k^{0(1)} = k^{0(1)}(0, \vec{\tau}_1) + \int_0^{\tau_0} d\lambda \int d\mathbf{x}_1 \frac{\partial \phi}{\partial \vec{x}}(\vec{x} - \vec{x}_1) \frac{\partial k^{1(0)}}{\partial \vec{v}}(\vec{x} - \vec{x}_1 - \vec{v}\lambda, \vec{v}, 0, \vec{\tau}_1) \quad (2.59)$$

Now, we examine the case $s = 1$ and we have

$$\begin{aligned} \frac{\partial k^{1(1)}}{\partial \tau_0} + \frac{\partial k^{1(0)}}{\partial \tau_1} + \vec{v} \cdot \vec{\nabla} k^{1(1)} &= \frac{\partial \phi}{\partial \vec{x}}(\vec{x} - \vec{x}_1) \cdot \frac{\partial k^{0(0)}}{\partial \vec{v}} + \frac{\partial \phi}{\partial \vec{x}}(\vec{x} - \vec{x}_1) \cdot \frac{\partial k^{1(0)}}{\partial \vec{v}} + \\ &+ \int d\vec{x}_2 \frac{\partial \phi}{\partial \vec{x}}(\vec{x} - \vec{x}_2) \frac{\partial k^{2(0)}}{\partial \vec{v}}(\xi_1 - v\tau_0, \xi_2 - v\tau_0, v, 0, \vec{\tau}_1) \end{aligned} \quad (2.60)$$

Integration yields

$$\begin{aligned} k^{1(1)} &= k^{1(1)}(x - x_1 - v\tau_0, v, 0, \vec{\tau}_1) + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial \vec{x}}(x - x_1 - v\lambda) \frac{\partial k^{0(0)}}{\partial \vec{v}}(\tau_2) + \\ &+ \int_0^{\tau} d\lambda \frac{\partial \phi}{\partial \vec{x}}(x - x_1 - v\lambda) \frac{\partial^{(\lambda)} k^{1(0)}}{\partial \vec{v}}(x - x_1 - v\tau_0, v, 0, \vec{\tau}_1) + \\ &+ \int_0^{\tau_0} d\lambda \int d\mathbf{x}_2 \frac{\partial \phi}{\partial \vec{x}}(x - x_2 - v\lambda) \frac{\partial^{(\lambda)} k^{2(0)}}{\partial \vec{v}}(x - x_1 - v\tau_0, x - x_2 - v\tau_0, v, 0, \vec{\tau}_1) \\ &- \frac{\partial}{\partial \tau_1} \tau_0 k^{1(0)}(x - x_1 - v\tau_0, v, 0, \vec{\tau}_1) \end{aligned} \quad (2.61)$$

The last term would be a pseudo-secularity and for large $x - x_1$

would produce larger values than the leading-order term. Therefore, we set

$$\frac{\partial k^1(0)}{\partial \tau_1} = 0 \quad (2.62)$$

and we conclude

$$k^1(0) = k^1(0)(x-x_1-v\tau_0, v_1 0, 0, \vec{\tau}_2) \quad (2.63)$$

which replaces Eq. (2.54). We are then left with the following quantity

$$\begin{aligned} k^1(1) = & k^1(1)(x-x_1-v\tau_0, v, 0, \vec{\tau}_1) + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \cdot \frac{\partial k^0(0)}{\partial v}(\vec{\tau}_2) + \\ & + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \cdot \frac{\partial^{(\lambda)} k^1(0)}{\partial v}(x-x_1-v\tau_0, v, 0, \vec{\tau}_1) + \\ & + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial^{(\lambda)} k^2(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, \vec{\tau}_1) \end{aligned} \quad (2.64)$$

Now, we examine the case $s = 2$. The equation is given by

$$\begin{aligned} \frac{\partial k^2(1)}{\partial \tau_0} + \frac{\partial k^2(0)}{\partial \tau_1} + v \cdot \nabla k^2(1) = & \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k_2^1(0)}{\partial v} + \frac{\partial \phi}{\partial x}(x-x_2) \cdot \frac{\partial k_1^1(0)}{\partial v} + \\ & + \frac{\partial \phi}{\partial x}(x-x_2) \frac{\partial k_{12}^2(0)}{\partial v} + \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k_{12}^2(0)}{\partial v} + \int dx_3 \frac{\partial \phi}{\partial x}(x-x_3) \cdot \frac{\partial k_{123}^3(0)}{\partial v} \end{aligned} \quad (2.65)$$

Integration yields

$$\begin{aligned}
k^{2(1)} &= k^{2(1)}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, \vec{\tau}_1) + \\
&+ \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial^{(\lambda)} k^1(0)}{\partial v}(x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \right. \\
&+ \left. \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial^{(\lambda)} k^1(0)}{\partial v}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \right] + \\
&+ \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) + \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \right] \cdot \\
&\cdot \frac{\partial^{(\lambda)} k^2(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, \vec{\tau}_2) + \\
&+ \int_0^{\tau_0} d\lambda \int dx_3 \frac{\partial \phi}{\partial x}(x-x_3-v\lambda) \frac{\partial^{(\lambda)} k^3(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, \\
&\quad x-x_3-v\tau_0, v, 0, \vec{\tau}_1) + \\
&- \tau_0 \frac{\partial}{\partial \tau_1} k^{2(0)}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, \vec{\tau}_1) \tag{2.66}
\end{aligned}$$

By the previous argument, we set

$$\frac{\partial k^{2(0)}}{\partial \tau_1} = 0 \tag{2.67}$$

so that we conclude

$$k^{2(0)} = k^{2(0)}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) \tag{2.68}$$

This equation replaces Eq. (2.54) once more. Returning to (2.66), we are left with

$$\begin{aligned}
k^2(1) = & k^2(1)(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_1) + \\
& + \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \right. \\
& + \left. \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \right] + \\
& + \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) + \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \right] \frac{\partial(\lambda)_k^2(0)}{\partial v} (x-x_1-v\tau_0, \\
& \qquad \qquad \qquad x-x_2-v\tau_0, v, 0, \vec{\tau}_1) \\
& + \int_0^{\tau_0} d\lambda \int dx_3 \frac{\partial \phi}{\partial x}(x-x_3-v\lambda) \frac{\partial(\lambda)_k^3(0)}{\partial v} (x-x_1-v\tau_0, x-x_2-v\tau_0, \\
& \qquad \qquad \qquad x-x_3-v\tau_0, v, 0, \vec{\tau}_1) \quad (2.69)
\end{aligned}$$

In general, we can write

$$\begin{aligned}
k^s(1) = & k^s(1)(x-x_j-v\tau_0, v, 0, 0, \vec{\tau}_1) + \\
& + \sum_{i=1}^s \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_i-v\lambda) \frac{\partial(\lambda)_k^{s-1}(0)}{\partial v} (x-x_{(1)}-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\
& + \sum_{i=1}^s \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_i-v\lambda) \frac{\partial(\lambda)_k^s(0)}{\partial v} (x-x_2-v\tau_0, v, 0, \vec{\tau}_1) + \\
& + \int_0^{\tau_0} d\lambda \int dx_{s+1} \frac{\partial \phi}{\partial x}(x-x_{s+1}-v\lambda) \frac{\partial(\lambda)_k^{s+1}(0)}{\partial v} (x-x_j-v\tau_0, v, 0, \vec{\tau}_1) \\
& \qquad \qquad \qquad (2.70)
\end{aligned}$$

By the previous arguments, we write

$$\frac{\partial k^s(0)}{\partial \tau_1} = 0 \quad (2.71)$$

so that we conclude

$$k^s(0) = k^s(0)(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \quad (2.72)$$

Therefore, we conclude that Eq. (2.54) has been replaced by Eq. (2.72) for all values of s .

Now, we consider the second-order theory. First, we consider $s = 0$. The equation is given by

$$\begin{aligned} \frac{\partial k^{0(2)}}{\partial \tau_0} + \frac{\partial k^{0(1)}}{\partial \tau_1} + \frac{\partial k^{0(0)}}{\partial \tau_2} &= \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{1(1)}}{\partial v} = \\ &= \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \left[k^{1(1)}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_1) + \right. \\ &+ \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) + \\ &+ \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial^{(\lambda)} k^{1(0)}}{\partial v}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\ &+ \left. \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial}{\partial x}(x-x_2-v\lambda) \frac{\partial^{(\lambda)} k^{2(0)}}{\partial v} \right. \\ &\left. \cdot (x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) \right] \end{aligned} \quad (2.73)$$

Integration yields

$$\begin{aligned} k^{0(2)} &= \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{1(1)}}{\partial v}(x-x_1-v\lambda, v, 0, 0, \vec{\tau}_1) + \\ &+ \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\infty d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) + \\ &- \int_0^\tau d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) + \end{aligned}$$

$$\begin{aligned}
& + \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \int_0^\lambda d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial (\mu)_k^1(0)}{\partial v} \cdot \\
& \cdot (x-x_1-v\lambda, v, 0, 0, \vec{\tau}_2) \\
& + \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \int_0^\lambda d\mu \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\mu) \frac{\partial (\mu)_k^2(0)}{\partial v} \cdot \\
& \cdot (x-x_1-v\lambda, x-x_2-v\lambda, v, 0, 0, \vec{\tau}_2) + \\
& - \tau_0 \frac{\partial k^{0(1)}}{\partial \tau_1} (0, \vec{\tau}_1) - \tau_0 \frac{\partial k^{0(0)}}{\partial \tau_2} (\vec{\tau}_2) \tag{2.74}
\end{aligned}$$

The second term can be written

$$\tau_0 \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\infty d\mu \frac{\partial \phi}{\partial x}(\xi_1-v\mu) \frac{\partial k^{0(0)}}{\partial v} (\vec{\tau}_2) \tag{2.75}$$

Other terms for large τ_0 behave as follows

$$\begin{aligned}
k^{0(2)} & \sim \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{1(1)}}{\partial v}(x-x_1-v\lambda, v, 0, \vec{\tau}_1) + \\
& - \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^{\tau_0} d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \cdot \\
& \cdot \frac{\partial k^{0(0)}}{\partial v} (\vec{\tau}_2) \\
& + \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^{\tau_0} d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \cdot
\end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\partial(\lambda)_k^{1(0)}}{\partial v}(x-x_1-v\lambda, v, 0, 0, \vec{\tau}_2) \\
& + \int_0^{\tau_0 \max} d\lambda \int dx_1 \frac{\partial\phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^{\tau_0 \max} d\mu \int dx_2 \frac{\partial\phi}{\partial x} \cdot \\
& \cdot (x-x_2-v\mu) \frac{\partial(\lambda)_k^{2(0)}}{\partial v}(x-x_1-v\lambda, x-x_2-v\lambda, v, 0, 0, \vec{\tau}_2) \quad (2.76)
\end{aligned}$$

Since, in each case, the quantity

$$\phi(x-x_1) \quad (2.77)$$

nails down x_1 which nails down λ , and so on, there is no secularity. We then conclude

$$\begin{aligned}
& - \frac{\partial k^{0(1)}}{\partial \tau_1}(0, \vec{\tau}_1) - \frac{\partial k^{0(0)}}{\partial \tau_2}(\vec{\tau}_2) + \int dx_1 \frac{\partial\phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\infty d\mu \cdot \\
& \cdot \frac{\partial\phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) = 0 \quad (2.78)
\end{aligned}$$

Integration of Eq. (2.78) with respect to τ_1 yields

$$\begin{aligned}
& -k^{0(1)}(0, \vec{\tau}_1) - \tau_1 \left(\frac{\partial k^{0(0)}}{\partial \tau_2}(\vec{\tau}_2) - \int dx_1 \frac{\partial\phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \cdot \right. \\
& \left. \cdot \int_0^\infty d\mu \frac{\partial\phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) \right) = 0 \quad (2.79)
\end{aligned}$$

Unless the bracket is zero, the function $k^{0(1)}(0, \tau_1)$ is secular which would make $k^{0(1)}$ itself secular because the second term in Eq. (2.59) is not a function of τ_1 and therefore cannot cancel the secular τ_1 dependence. If the bracket is zero,

$$\frac{\partial k^{0(1)}}{\partial \tau_1} (0, \vec{\tau}_1) = 0 \quad (2.80)$$

from which we conclude

$$k^{0(1)}(0, \vec{\tau}_1) = k^{0(1)}(0, 0, \vec{\tau}_2) \quad (2.81)$$

and also

$$k^{0(1)}(\vec{\tau}_0) = k^{0(1)}(\vec{\tau}_0, 0, \vec{\tau}_2) \quad (2.82)$$

The bracket is, in fact, the Fokker-Planck equation

$$\frac{\partial k^{0(0)}}{\partial \tau_2}(\vec{\tau}_2) = \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\infty d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) \quad (2.83)$$

and the function $k^{0(2)}$ is then given by

$$\begin{aligned} k^{0(2)} = & \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{1(1)}}{\partial v}(x-x_1-v\lambda, v, 0, \vec{\tau}_1) + \\ & - \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_\lambda^\infty d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) + \\ & + \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\lambda d\mu \frac{\partial \phi}{\partial x}(x-x_1-v\mu) \frac{\partial (\mu)_k^{1(0)}}{\partial v} \cdot \\ & \cdot (x-x_1-v\lambda, v, 0, 0, \vec{\tau}_2) \\ & + \int_0^{\tau_0} d\lambda \int dx_1 \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \int_0^\lambda d\mu \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\mu) \cdot \\ & \cdot \frac{\partial (\mu)_k^{2(0)}}{\partial v}(x-x_1-v\lambda, x-x_2-v\lambda, v, 0, 0, \vec{\tau}_2) \end{aligned} \quad (2.84)$$

We now examine the case $s = 1$. The equation is given by

$$\begin{aligned} \frac{\partial k^{1(2)}}{\partial \tau_0} + \frac{\partial k^{1(1)}}{\partial \tau_1} + \frac{\partial k^{1(0)}}{\partial \tau_2} + \vec{v} \cdot \vec{\nabla}_k k^{1(2)} &= \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{0(1)}}{\partial v} + \\ &+ \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial k^{1(1)}}{\partial v} + \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2) \frac{\partial k^{2(1)}}{\partial v} \end{aligned} \quad (2.85)$$

Upon substitution, this equation can be written

$$\begin{aligned} \frac{\partial k^{1(2)}}{\partial \tau_0} + \frac{\partial k^{1(1)}}{\partial \tau_1}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_1) + \frac{\partial k^{1(0)}}{\partial \tau_2}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\ + \vec{v} \cdot \vec{\nabla}_k k^{1(2)} = \\ = \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \left[k^{0(1)}(0, 0, \vec{\tau}_2) + \int_0^{\tau_0} d\lambda \int dx'_1 \frac{\partial \phi}{\partial x}(x-x'_1) \cdot \right. \\ \cdot \left. \frac{\partial k^{1(0)}}{\partial v}(x-x'_1-v\lambda+v, 0, 0, \vec{\tau}_2) \right] + \\ + \frac{\partial \phi}{\partial x}(x-x_1) \frac{\partial}{\partial v} \left\{ k^{1(1)}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_1) + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \cdot \right. \\ \cdot \left. \frac{\partial k^{0(0)}}{\partial v}(\vec{\tau}_2) + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial^{(\lambda)} k^{1(0)}}{\partial v} \cdot \right. \\ \cdot (x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\ + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial^{(\lambda)} k^{2(0)}}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) \Big\} \\ + \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2) \frac{\partial}{\partial v} \left\{ k^{2(1)}(x-x_1-v\tau_0, x-x_2-v\tau_0, 0, 0, \vec{\tau}_1) + \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (x-x_2-v\tau, v, 0, 0, \vec{\tau}_2) + \right. \\
& + \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (x-x_1-v\tau, 0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \left[\frac{\partial \phi}{\partial x}(x-x_1-v\lambda) + \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \right] \frac{\partial(\lambda)_k^2(0)}{\partial v} \cdot \\
& \cdot (x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \int dx_3 \frac{\partial \phi}{\partial x}(x-x_3-v\lambda) \frac{\partial(\lambda)_k^3(0)}{\partial v} \cdot \\
& \cdot (x-x_1-v\tau_0, x-x_2-v\tau_0, x-x_3-v\tau_0, v, 0, 0, \vec{\tau}_2) \Big\} \quad (2.86)
\end{aligned}$$

Performing the τ_0 integration yields

$$\begin{aligned}
k^1(2) &= \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)_k^0(1)}{\partial v} (0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \int_{\partial k} dx_1 \frac{\partial \phi}{\partial x}(x-x_1'-v\lambda) \cdot \\
& \cdot \frac{\partial(\lambda)_k^1(0)}{\partial v} (x-x_1'-v(\lambda+\mu), v, 0, 0, \tau_1) \cdot \\
& \cdot \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)}{\partial v} k^1(1)(x-x_1-v\tau_0, v, 0, \vec{\tau}_1) + \\
& + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \frac{\partial \phi}{\partial x}(x-x_1-v(\lambda+\mu)) \frac{\partial k^2(0)}{\partial v} (\tau_0) \\
& + \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \frac{\partial \phi}{\partial x}(x-x_1-v(\lambda+\mu)) \cdot
\end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\partial(\mu)_k^1(0)}{\partial v}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \frac{\partial\phi}{\partial x}(x-x_1-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \int dx_2 \frac{\partial\phi}{\partial x}(x-x_2-v(\lambda+\mu)) \cdot \\
& \cdot \frac{\partial(\lambda)_k^2(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial\phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)_k^2(1)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) \\
& + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial\phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \left[\frac{\partial\phi}{\partial x}(x-x_1-v(\lambda+\mu)) \right] \cdot \\
& \cdot \frac{\partial(\lambda+\mu)_k^1(0)}{\partial v}(x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \frac{\partial\phi}{\partial x}(x-x_2-v(\lambda+\mu)) \cdot \\
& \cdot \frac{\partial(\lambda+\mu)_k^1(0)}{\partial v}(x-x_1-v\tau_0) \\
& + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial\phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \left[\frac{\partial\phi}{\partial x}(x-x_1-v(\lambda+\mu)) + \right. \\
& + \left. \frac{\partial\phi}{\partial x}(x-x_2-v(\lambda+\mu)) \right] \frac{\partial(\lambda+\mu)_k^2(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\
& + \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial\phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\tau} d\mu \int dx_3 \frac{\partial\phi}{\partial x} \cdot \\
& \cdot (x-x_3-v(\lambda+\mu)) \frac{\partial(\lambda+\mu)_k^3(0)}{\partial v}(x-x_1-v\tau_0, x-x_2-v\tau_0, x-x_3-v\tau_0, v, 0, 0, \vec{\tau}_2) \\
& - \tau_0 \left(\frac{\partial k^1(1)}{\partial \tau_1}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_1) + \frac{\partial k^1(0)}{\partial \tau_2}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \right)
\end{aligned}$$

(2.87)

The term containing the star is a pseudo-secularity; in fact, we have

$$\int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^{\tau_0-\lambda} d\mu \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v(\lambda+\mu)) \frac{\partial(\lambda+\mu)_k^1(0)}{\partial v} (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \quad (2.88)$$

This seems to have secularities proportional to various powers of τ_0 , in particular, quadratic and cubic.

We examine in detail the following integral

$$\begin{aligned} & \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2) \frac{\partial}{\partial v} \int_0^{\tau_0} d\lambda \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) = \\ & = \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2) \frac{\partial}{\partial v} \int_0^{\infty} d\lambda \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) + \\ & - \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2) \frac{\partial}{\partial v} \int_{\tau_0}^{\infty} d\lambda \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)_k^1(0)}{\partial v} (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \end{aligned} \quad (2.89)$$

We study the second term of Eq. (2.89) written as

$$\begin{aligned} & - \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_{\tau_0-\lambda}^{\infty} d\mu \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v(\lambda+\mu)) \cdot \\ & \cdot \frac{\partial(\lambda+\mu)_k^1(0)}{\partial v} (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \\ & - \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_{\tau_0}^{\infty} dz \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_2-vz) \frac{\partial(z)_k^1(0)}{\partial v} \cdot \\ & \cdot (\mathbf{x}-\mathbf{x}_1-v\tau_0, v, 0, 0, \vec{\tau}_2) \end{aligned} \quad (2.90)$$

Upon streaming, the first term of Eq. (2.1) is given by

$$\begin{aligned}
& \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_0^\infty d\mu \frac{\partial \phi}{\partial x}(x-x_2-v(\lambda+\mu)) \frac{\partial(\lambda+\mu)_k^1(0)}{\partial v} \cdot \\
& \quad \cdot (x-x_1-v\tau_0, v, 0, 0, \frac{1}{\tau_2}) \\
& = \int_0^{\tau_0} d\lambda \int dx_2 \frac{\partial \phi}{\partial x}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v} \int_\lambda^\infty dz \frac{\partial \phi}{\partial x}(x-x_2-vz) \frac{\partial(\lambda)_k^1(0)}{\partial v} \cdot \\
& \quad \cdot (x-x_1-v\tau_0, v, 0, 0, \frac{1}{\tau_2}) \tag{2.91}
\end{aligned}$$

We can evaluate the collisional part of the integral as follows

$$\begin{aligned}
& \int dx_2 \frac{\partial \phi}{\partial x_1}(x-x_2) \frac{\partial}{\partial v_1} \int_0^\infty d\lambda \frac{\partial \phi}{\partial x_j}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v_j} = \\
& = \frac{\partial}{\partial v_1} \int_0^\infty d\lambda \left\{ \int dx_2 \frac{\partial \phi}{\partial x_1}(x-x_2) \frac{\partial \phi}{\partial x_j}(x-x_2-v\lambda) \frac{\partial(\lambda)}{\partial v_j} \right\} \tag{2.92}
\end{aligned}$$

We now introduce variables as

$$\text{let } x - x_2 = x', \quad v\lambda = \mu \tag{2.93}$$

We can write the tensor part of this equation as

$$\begin{aligned}
T_{1j}(\mu) &= \int dx \frac{\partial \phi(x)}{\partial x_1} \frac{\partial \phi(x-\mu)}{\partial x_j} = - \int dx \phi(x) \frac{\partial^2 \phi(x-\mu)}{\partial x_1 \partial x_j} = \\
&= - \int dx \phi(x) \frac{\partial^2 \phi(x-\mu)}{\partial \mu_1 \partial \mu_j} \\
&= - \frac{\partial^2}{\partial \mu_1 \partial \mu_j} \int dx \phi(x) \phi(x-\mu) = - \frac{\partial^2}{\partial \mu_1 \partial \mu_j} \Omega(\mu) = - \frac{\partial}{\partial \mu_1} e_j \Omega'(\mu) \\
&=
\end{aligned}$$

$$= -\left(\frac{\delta_{1j}-e_1e_j}{e^2}\right) \frac{\Omega'(\mu)}{\mu} - e_je_1\Omega''(\mu) = T_{1j}(\mu) \quad (2.94)$$

and we now have

$$\begin{aligned} \int_0^\infty d\lambda \, T_{1j}(\mu) \frac{\partial}{\partial v_j} &= \frac{1}{v} \int_0^\infty d\mu \, T_{1j}(\mu) \frac{\partial}{\partial v_j} = \\ &= \left\{ -\frac{\delta_{1j}+e_1e_j}{v} \int_0^\infty d\mu \, \frac{\Omega'(\mu)}{\mu} - \frac{e_1e_j}{v} \int_0^\infty d\mu \, \Omega''(\mu) \right\} \frac{\partial}{\partial v_j} = -\frac{(\delta_{1j}-e_1e_j)}{v} \cdot \\ &\cdot \int_0^\infty d\mu \, \frac{\Omega'(\mu)}{\mu} \frac{\partial}{\partial v_j} \end{aligned} \quad (2.95)$$

We can write this as

$$\begin{aligned} \int_0^\infty \lambda d\lambda T_{1j}(\mu) \frac{\partial}{\partial x_j} &= \frac{1}{v^2} \int_0^\infty \mu d\mu T_{1j}(\mu) \frac{\partial}{\partial x_j} = \\ &= \left(-\frac{(\delta_{1j}-e_1e_j)}{v^2} \int_0^\infty d\mu \Omega'(\mu) - \frac{e_1e_j}{v^2} \int_0^\infty \mu d\mu \Omega''(\mu) \right) \frac{\partial}{\partial x_j} \\ &= \left(\frac{(\delta_{1j}-e_1e_j)}{v^2} \Omega(0) + \frac{e_1e_j}{v^2} \int_0^\infty d\mu \Omega'(\mu) \right) \frac{\partial}{\partial x_j} \\ &= \left(\frac{(\delta_{1j}-e_1e_j)}{v^2} \Omega(0) - \frac{e_1e_j}{v^2} \Omega(0) \right) \frac{\partial}{\partial x_j} \\ &= \frac{(\delta_{1j}-2e_1e_j)}{v^2} \Omega(0) \frac{\partial}{\partial x_j} \end{aligned} \quad (2.96)$$

This leads to the following expression

$$\begin{aligned} \frac{\partial k^1(0)}{\partial \tau_2}(x-x_1-v\tau_0, v, 0, 0, \vec{\tau}_2) = & \left\{ -\alpha \frac{\partial}{\partial v_1} \left(\frac{(\delta_{1j} - e_1 e_j)}{v} \right) \frac{\partial}{\partial v_j} + \right. \\ & \left. + \Omega(0) \frac{\partial}{\partial v_1} \left(\frac{\delta_{1j} - 2e_1 e_j}{v^2} \right) \frac{\partial}{\partial x_j} \right\} k^1(0) \quad (2.97) \end{aligned}$$

We now investigate

$$\begin{aligned} \alpha &= \int_0^\infty d\mu \frac{\Omega'(\mu)}{\mu} = \int_0^\infty \frac{d\mu}{\mu} \frac{\partial}{\partial \mu} \int dx \phi(x) \phi(x-\mu) = \\ &= (2\pi)^3 \int_0^\infty \frac{d\mu}{\mu} \frac{d}{d\mu} \int dk |\tilde{\phi}(k)|^2 e^{-ik\mu} = (2\pi)^3 \int_0^\infty \frac{d\mu}{\mu} \frac{d}{d\mu} \int dk |\tilde{\phi}|^2 e^{ik \cdot e \mu} \\ &= (2\pi)^3 \int_0^\infty \frac{d\mu}{\mu} \int d\vec{k} |\tilde{\phi}|^2 (-i\vec{k} \cdot \vec{e}) e^{-ik\mu} = (2\pi)^3 \frac{1}{2} \int_{-\infty}^\infty \frac{d\mu}{\mu} \int dk |\tilde{\phi}|^2 \cdot \\ &\quad \cdot (ik \cdot e) e^{-ik\mu} \quad (2.98) \end{aligned}$$

Now,

$$J(\alpha) = \int_{-\infty}^\infty \frac{d\mu}{\mu} e^{i\alpha\mu}, \quad \frac{\partial J}{\partial \alpha} = i \int_{-\infty}^\infty d\mu e^{i\alpha\mu} = 2\pi i \delta(\alpha) \quad (2.99)$$

Notice that J is an odd integral. Therefore, we conclude

$$\begin{aligned} J(\alpha) &= \pi i \epsilon(\alpha) \\ \epsilon &= 1 \quad \alpha > 0 \\ \epsilon &= -1 \quad \alpha < 0 \end{aligned} \quad (2.100)$$

Therefore, we obtain for α

$$\begin{aligned} &\frac{(2\pi)^3}{2} \int dk |\tilde{\phi}|^2 (-ik \cdot e) \pi i \epsilon(-k \cdot e) = \\ &= - \frac{(2\pi)^3}{2} \pi \int d\vec{k} |\tilde{\phi}|^2 k \cdot e \epsilon(k \cdot e) \quad (2.101) \end{aligned}$$

We have the identity

$$x \epsilon(x) = |x| \quad (2.102)$$

Therefore, we have

$$\alpha = -4\pi^4 \int d\vec{k} |\vec{\phi}|^2 |\vec{k} \cdot \mathbf{e}| < 0 \quad (2.103)$$

We can then rewrite Eq. (2.97) as

$$\frac{\partial k^1(0)}{\partial \tau_2} = \left\{ |\alpha| \frac{\partial}{\partial v_1} \frac{(\delta_{1j} - e_1 e_j)}{v} \frac{\partial}{\partial v_j} + |\Omega(0)| \frac{\partial}{\partial v_1} \left(\frac{\delta_{1j} - 2e_1 e_j}{v^2} \right) \frac{\partial}{\partial x_j} \right\} k^1(0) \quad (2.104)$$

To test for positive-definite nature of the collision integral, we compute

$$\begin{aligned} (\phi, A \phi) &= \iint dx dv \phi(x-vt, v) A \phi(x-vt) \\ &= -|\alpha| \iint dx dv \frac{\partial \phi}{\partial v_1} \frac{(\delta_{1j} - e_1 e_j)}{v} \frac{\partial \phi}{\partial v_j} \\ &\quad - |\Omega(0)| \iint dx dv \left[\frac{\partial}{\partial v_1} \left[\phi(x-v\tau_0, v) \right] \frac{\delta_{1j} - 2e_1 e_j}{v^2} \left[\frac{\partial}{\partial x_j} \phi(x-v\tau_0, v) \right] \right] \\ &= -|\Omega(0)| \iint dx dv \frac{\partial}{\partial v_1} \left(e^{-v \cdot \nabla \tau_0} \phi(x, v) \right) \frac{\delta_{1j} - 2e_1 e_j}{v^2} \frac{\partial}{\partial x_j} e^{-v \cdot \nabla \tau_0} \phi(x, v) \\ &= -|\Omega(0)| \iint dx dv \left[e^{-v \cdot \nabla \tau_0} \left(\frac{\partial}{\partial v_1} - \tau_0 \frac{\partial}{\partial x_1} \right) \phi(x, v) \right] \frac{\delta_{1j} - 2e_1 e_j}{v^2} \cdot \\ &\quad \cdot \left(\frac{\partial}{\partial x_j} e^{-v \cdot \nabla \tau_0} \phi(x, v) \right) \\ &= -|\Omega(0)| \iint dx dv \left(\frac{\partial}{\partial v_1} \phi(x, v) \right) \left(\frac{\delta_{1j} - 2e_1 e_j}{v^2} \right) \frac{\partial}{\partial x_j} \phi(x, v) \\ &\quad + |\Omega(0)| \tau_0 \iint dx dv \left[\frac{\partial}{\partial x_1} \phi(x, v) \right] \left(\frac{\delta_{1j} - 2e_1 e_j}{v^2} \right) \frac{\partial \phi}{\partial x_j}(x, v) \end{aligned} \quad (2.105)$$

The alpha term can be written

$$\begin{aligned}
& -|\alpha| \iint dx dv \left[\left(\frac{\partial}{\partial v_1} - \tau_0 \frac{\partial}{\partial x_1} \right) \right] \phi(x, v) \frac{\delta_{1j}^{-e_1 e_j}}{v} \left[\left(\frac{\partial}{\partial v_j} - \tau_0 \frac{\partial}{\partial x_j} \right) \phi(x, v) \right] = \\
& = -|\alpha| \iint dx dv \frac{\partial \phi}{\partial v_1} \frac{\delta_{1j}^{-e_1 e_j}}{v} \frac{\partial \phi}{\partial v_j} + 2|\alpha| \tau_0 \iint dx dv \frac{\partial \phi}{\partial x_1} \frac{\delta_{1j}^{-e_1 e_j}}{v} \frac{\partial \phi}{\partial v_j} \\
& -|\alpha| \tau_0^2 \iint dx dv \frac{\partial \phi}{\partial x_1} \frac{\delta_{1j}^{-e_1 e_j}}{v} \frac{\partial \phi}{\partial x_1} \tag{2.106}
\end{aligned}$$

We see that for large τ_0 the τ_0^2 term dominates.

III. NONADIABATIC ANALYSIS

By considering the Lorentz gas in a weakly coupled regime, we have shown that a correct lowest-order two-particle correlation function solution to the BBGKY hierarchy is given via a Green's function whose Fourier transform satisfies the analyticity requirements for causality. The Bogolubov adiabatic approximation is shown to imply a solution which is acausal. Extension to the s-particle correlation function is discussed.

The successful derivation of dispersion relations for the fundamental correlation function (Green's function) gives us conditions that any approximation to the behavior of a plasma must satisfy, including instabilities.

The Lorentz gas, consisting of one electron and N-1 ions, is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \sum_{i=1}^{N-1} \phi(\vec{x} - \vec{x}_i) = \frac{p^2}{2} + \sum_{i=1}^{N-1} \phi_i(\vec{x}) \quad (3.1)$$

where the ions have fixed positions, \vec{x}_i , and the electron has unit mass. The BBGKY hierarchy for the s-body distribution function, F^s , for this system reads

$$\begin{aligned} \frac{\partial F^s}{\partial t} + \vec{v} \cdot \vec{\nabla} F^s &= \frac{\phi_0}{kT} \left(\vec{\nabla} \sum_{i=1}^{s-1} \phi_i \right) \cdot \vec{\nabla}_v F^s + \\ &+ \left(\frac{\phi_0}{kT} \right) (nr_0^3) \int d\vec{x}_s \vec{\nabla} \phi_s \cdot \vec{\nabla}_v F^{s+1} \end{aligned} \quad (3.2)$$

where the magnitude of the potential, ϕ_0 , and its range, r_0 , are introduced so that ϕ_i has unit magnitude and range. In the weakly-coupled gas regime,

$$\frac{\phi_0}{kT} = \epsilon \ll 1 \quad \text{and} \quad nr_0^3 = 1 \quad (3.3)$$

Spatial homogeneity of the electron distribution gives

$$F^1 = F(\vec{v}, t) \quad (3.4)$$

and for spatially homogeneous and uncorrelated ion distributions,

$$F^2 = F + G(\vec{x}, \vec{x}_1, \vec{v}, t) \quad (3.5)$$

where G is the two-body correlation function. Substitution into the F^1, F^2 equations implies

$$\frac{\partial F}{\partial t} = \epsilon \int d\vec{x}_1 \vec{\nabla} \phi_1 \cdot \vec{\nabla}_v G \quad (3.6)$$

$$\frac{\partial G}{\partial t} + \vec{v} \cdot \vec{\nabla} G = \epsilon \vec{\nabla} \phi_1 \cdot \vec{\nabla}_v F \quad (3.7)$$

where integral terms of order ϵ involving F have been neglected in (3.6) and ones involving $G(\vec{x}, \vec{x}_2)$ and $H(\vec{x}, \vec{x}_1, \vec{x}_2)$ (the three-body correlation function) have been neglected in (3.7). Hereafter, \vec{x}_1 is taken to be zero except when several ions are considered.

The fact that the equations are linear implies that they can be regarded as describing a superposition of equilibrium and nonequilibrium terms

$$F = F_{eq} + f, \quad G = G_{eq} + g \quad (3.8)$$

where the equilibrium terms must satisfy

$$0 = \int d\vec{x} \vec{\nabla} \phi \cdot \vec{\nabla}_v G_{eq} \quad (3.9)$$

$$\vec{v} \cdot \vec{\nabla} G_{eq} = \epsilon \vec{\nabla} \phi \cdot \vec{\nabla}_v F_{eq}$$

Any spherically symmetric function for $F_{eq}(\vec{v})$ leads to stationary solutions for these equations and the equilibrium terms in Eqs. (3.6) and (3.7) can then be dropped. The nonequilibrium terms satisfy

$$\frac{\partial F}{\partial t} = \epsilon \int d\vec{x} \vec{\nabla} \phi \cdot \vec{\nabla}_v g \quad (3.6')$$

$$\frac{\partial \vec{g}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{g} = \epsilon \vec{\nabla} \phi \cdot \vec{\nabla}_v \quad (3.7')$$

Fourier transform methods on (3.7') readily yield

$$\begin{aligned} g(\vec{x}, \vec{x}_1, \vec{v}, t) &= \epsilon \int_0^t dt' \int d\vec{x}' \delta(\vec{x} - \vec{x}' - \vec{v}(t-t')) \vec{\nabla} \phi \cdot \vec{\nabla}_v f(t) \\ &= \epsilon \int_0^t d\lambda \vec{\nabla} \phi(\vec{x} - \vec{v}\lambda) \cdot \vec{\nabla}_v f(t-\lambda) \end{aligned} \quad (3.10)$$

where it is assumed that $g(t < 0) = 0$; i.e., that the system is disturbed from equilibrium at $t = 0$.

Substitution of (3.10) into (3.6') yields

$$\frac{\partial f}{\partial t} = \epsilon^2 \int_0^t d\lambda \int d\vec{x} \vec{\nabla} \phi \cdot \vec{\nabla}_v (\vec{\nabla} \phi(\vec{x} - \vec{v}\lambda) \cdot \vec{\nabla}_v f(t-\lambda)) \quad (3.11)$$

an integrodifferential equation in \vec{v} and t for f . Solution of this equation for appropriate initial conditions would give the lowest-order behavior of f for all time, following some initial disturbance from equilibrium.

Given such a solution, Eqs. (3.6') and (3.7') are decoupled and (3.10) is simply an integral expression for g which can be viewed as a Green's function solution to (3.7')

$$g(\vec{x}, \vec{x}_1, \vec{v}, t) = \int_{-\infty}^{\infty} dt' \int d\vec{x}' G(\vec{x} - \vec{x}', t - t') \underline{\epsilon \vec{\nabla} \phi \cdot \vec{\nabla}_v f(\vec{x}', \vec{v}, t')} \quad (3.12)$$

in terms of a correlation "source" (underlined) which is assumed to be zero for $t' < 0$. It is of some importance to note from Eq. (3.10) that G does not tamper with the form of the dependence on \vec{v} in the source; in fact, \vec{v} is basically a parameter in the relationship between g and its "source."

The standard one-dimensional analysis of causality proceeds from the inspection of the Fourier transform of the Green's function $G(\tau)$; Titchmarsh's theorem gives equivalence of the analyticity of

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G(\tau) \quad (3.13)$$

in the upper half plane and the requirement $G(\tau) = 0$ for $t < 0$. In the present case, the condition $G(\vec{x}, \tau) = 0$ for $\tau < x/v$ demands consideration of

$$A(\vec{k}, \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \int d\vec{x} e^{-i\vec{k} \cdot \vec{x}} G(\vec{x}, \tau) \quad (3.14)$$

which is, in fact, analytic in the upper half plane for G zero outside the forward cone $x_1 \leq v_1 \tau$.

The Green's function $G(\vec{x}, \tau) = \delta(\vec{x} - \vec{v}\tau) \theta(\tau)$ for the correlation is manifestly causal and

$$A(\vec{k}, \omega) = \frac{1}{(2\pi)^2} \int_0^{\infty} d\tau e^{-i(\vec{k} \cdot \vec{v} - \omega)\tau} \quad (\vec{k} \cdot \vec{v} \text{ real}) \quad (3.15)$$

Introducing $\omega = \omega_r + i\omega_1$,

$$A(\vec{k}, \omega) = \frac{1}{(2\pi)^2} \frac{1}{(\omega_r + i\omega_1) - \vec{k} \cdot \vec{v}} \quad \text{if } \omega_1 > 0 \quad (3.16)$$

i.e., $A(\vec{k}, \omega)$ is verified to be analytic in the upper half ω -plane. $A(\vec{k}, \omega)$ reduces to a constant times the distribution $\delta_+(\omega - \vec{k} \cdot \vec{v})$ on the real axis and also satisfies dispersion relations there.

The Bogolubov analysis of the correlation function, g_B , rests on the assumptions

- i) $g_B = \sum_{n=0}^{\infty} \epsilon^n g_n(\vec{x}, \vec{x}_1, \vec{v}, t)$
- ii) g_B depends on time only through f
- iii) $g_B(x - v\lambda, t) \xrightarrow[\lambda \rightarrow \infty]{} 0$.

Writing $\vec{s} = \epsilon \nabla_v f$ and noting that g_B depends on f only

through \vec{s} , it follows that

$$\frac{\partial g}{\partial t} = \frac{\delta g}{\delta \vec{s}} \cdot \frac{\partial \vec{s}}{\partial t} \quad (3.17)$$

where $\delta g / \delta \vec{s}$ is the vector whose components are functional derivatives of g with respect to components of \vec{s} .

Equation (3.7') with substitution from (3.6) then becomes

$$\vec{v} \cdot \vec{\nabla} g_B + \epsilon^2 \frac{\delta g_B}{\delta \vec{s}} \cdot \vec{\nabla}_v \left(\int d\vec{x} \vec{\nabla} \phi \cdot \vec{\nabla}_v g_B \right) = \vec{s} \cdot \vec{\nabla} \phi \quad (3.18)$$

Use of (i), equating powers, and use of (iii) implies:

$$g_0 = \frac{\vec{s}}{v} \cdot \int_{-\infty}^{x_{||}} \nabla \phi \, dx'_{||} \quad (x_{||} = \vec{x} \cdot \vec{v} / v)$$

$$g_n = 0 \quad \text{for } n \text{ odd}$$

$$g_2 = \frac{1}{v^2} \frac{\partial^2}{\partial v_1 \partial v_j} (x_{k/v}) \cdot T_{1jk} \quad , \quad \text{where } T \text{ depends on } \underline{\text{position only}} \quad (3.19)$$

and similar contractions of tensors depending on \vec{v}, t with ones depending on \vec{x}, \vec{x}_1 for higher-order even terms.

From the form of these solutions, it is evident that the time dependence of g is not of a retarded character but instantaneous. Consequently, the solution is acausal. Consideration of a Green's function and its Fourier transform is not appropriate for (3.18) since it is not linear in g_B but it is appropriate for

$$\vec{v} \cdot \vec{\nabla} g_B + \epsilon^2 \frac{\delta g_B}{\delta \vec{s}} \cdot \vec{\nabla}_v \frac{\partial f}{\partial t} = \vec{s} \cdot \vec{\nabla} \phi \quad (3.20)$$

where f and \vec{s} are assumed known. Expansion in powers of ϵ again gives the result that time dependence is instantaneous and thus $G_B(\vec{x}, \tau)$ for Eq. (3.20) must contain a factor $\delta(\tau)$.

As a consequence of this, $A_B(\vec{k}, \omega)$ is a constant function of ω and consideration of analyticity is pointless. In any case, the instantaneous G_B clearly is not causal in the same sense that the exact $G = \delta(\vec{x} - \vec{v}\tau)\theta(\tau)$ is. The adiabatic assumption (ii) is simply too crude an approximation to give the proper causality result.

Green's function analysis of the linear time scale solutions for g leads to

$$G_{LTS} = \delta(\vec{x} - \vec{x}' - \vec{v}(\tau_0 - \tau_0'))\theta(\tau_0 - \tau_0')\delta(\tau_2 - \tau_2') \quad (3.21)$$

through second order; clearly, the result is causal on the τ_0 scale and acausal on the τ_2 scale.

The results for the two-body correlation function $G(\vec{x}, \vec{x}_1, \vec{v}, t) = G_1$ can be extended to s bodies. Writing

$$\begin{aligned} F^s = & K_{1,2,\dots,s-1} + \sum_{\substack{\text{combinations of} \\ s-1 \text{ indices in} \\ s-2 \text{ places}}} K_{i_1,\dots,i_{s-2}} \\ & + \dots + \sum_{i=1}^{s-1} G_i + F \end{aligned} \quad (3.22)$$

it can be shown by induction that the s -body correlation function satisfies

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right] K_{1,2,\dots,s-1} = \text{source term involving lower-order correlations} \quad (3.23)$$

where terms of magnitude ϵ or smaller in the s -body and higher correlation functions have been dropped.

The exact Green's function, which is causal, again applies for the s -body correlation.

Example I - The Ionization Model

Consider the BBGKY hierarchy appropriate for a spatially homogeneous gas. In leading order for small momentum transfer ($\phi \ll kT = 1$), the two-particle correlation function, g , satisfies

$$\frac{\partial g}{\partial t} + \vec{v}_{12} \cdot \vec{\nabla}_{12} g = \frac{\partial \phi_{12}}{\partial \vec{x}_{12}} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \right) F_1 F_2 \quad (3.24)$$

F_1 is the one-particle distribution function for the i 'th species. Equation (3.24) has the form of a continuity equation in configuration space. This fact implies that the "correlation charge"

$$Q \equiv \int d\vec{x}_{12} g(\vec{x}_{12}, \vec{v}_1, \vec{v}_2, t) \quad (3.25)$$

can be expressed directly in terms of the source of g

$$Q = \int_0^t d\lambda \int d\vec{x}_{12} \frac{\partial \phi_{12}}{\partial \vec{x}_{12}} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \right) F_1(\vec{v}_1, \lambda) F_2(\vec{v}_2, \lambda) \quad (3.26)$$

Note that the definition of Q is introduced after the bulk limit ($N, V \rightarrow \infty$ with n fixed) has been taken (as implied by Eq. (3.24)).

Since $\int d\vec{x}_{12} (\partial \phi_{12} / \partial \vec{x}_{12}) = 0$ for any central potential, the total correlation charge vanishes in leading order

$$Q = 0 \quad (3.27)$$

Bogolubov's functional assumption violates this basic feature of the nonequilibrium equations. We illustrate this fact with a simple but instructive example. Consider a neutral, ideal gas in thermodynamic equilibrium. At $t = 0$, "ionize" the gas by switching on a two-body potential, ϕ_{12} . The solution of Eq. (3.24) is given by

$$g(\vec{x}_{12}, \vec{v}_1, \vec{v}_2, t) = -[\phi(\vec{x}_{12}) - \phi(\vec{x}_{12} - \vec{v}_{12} t)] M_1(\vec{v}_1) M_2(\vec{v}_2) \quad (3.28)$$

where $M(\vec{v})$ is the Maxwellian. Strictly speaking, Eq. (3.28) is

the result of linear time scale analysis. In a fully nonadiabatic treatment (G.V. Ramanathan, A.H. Kritz and G. Sandri, Phys. Lett. 31A, 477 (1970), the function M adjusts to the ionization. This adjustment does not alter qualitatively the considerations made here. From Eq. (3.28), we can prove directly that $Q = 0$ for all times. Indeed, from Eq. (3.28) we see that the correlation g has a charge localized at short interparticle separations equal in magnitude and opposite in sign to a correlation charge propagated in the direction of the relative velocity. The correlation function given by Eq. (3.28) does not reach thermodynamic equilibrium in a uniform sense. A proper description of the approach of the two-body correlation function to thermodynamic equilibrium must include three-body collisions that damp the travelling part of the correlation function. The travelling part of the correlation moves away from a fixed region of phase space so that

$$\lim_{\tau \rightarrow \infty} g(\vec{x}_{12}, \vec{v}_1, \vec{v}_2, t) = -\phi(\vec{x}_{12})M_1(\vec{v}_1)M_2(\vec{v}_2) \quad (3.29)$$

for fixed \vec{v}_{12} . However, we want to emphasize that Q is still zero, not the integral of the right-hand side of Eq. (3.29), $-\int \phi(\vec{x}_{12})M_1(\vec{v}_1)M_2(\vec{v}_2)d\vec{x}_{12}$.

Bogolubov's approximation for Eq. (3.24) is given by

$$g_B(\vec{x}_{12}, \vec{v}_1, \vec{v}_2, t) = \int_0^\infty d\lambda \frac{\partial \phi_{12}}{\partial \vec{x}_{12}}(\vec{x}_{12} - \vec{v}_{12}\lambda) \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \right) F_1(\vec{v}_1, t) F_2(\vec{v}_2, t) \quad (3.30)$$

which gives for the ionization model

$$g_B(\vec{x}_{12}, \vec{v}_1, \vec{v}_2, t) = -\phi(\vec{x}_{12})M_1(\vec{v}_1)M_2(\vec{v}_2) \quad (3.31)$$

(i.e., the "equilibrium result"). This result leads to a violation of Eq. (3.27) and shows that it is not safe to check a non-equilibrium result by inserting $F(\vec{v}, t) = M(\vec{v})$ and comparing the resulting expression with an equilibrium formula.

The failure of Bogolubov's method is more evident in the

presence of a strong magnetic field. In this case, the left-hand side of Eq. (3.24) is modified by the term $[q_1(\vec{v}_1 \times \vec{B}) \cdot \vec{\nabla}_{p_2} + q_2(\vec{v}_2 \times \vec{B}) \cdot \vec{\nabla}_{p_2}]g$. The ionization model gives

$$g = -[\phi(\vec{x}_{12}) - \phi(\vec{x}_{12} - \vec{v}_{12\parallel} t - \vec{v}_{12\perp} \sin Bt - \Omega \cdot \vec{v}_{12\perp} \cos Bt)] M_1 M_2 \quad (3.32)$$

where $\Omega_{ij} = \epsilon_{ijk} B_k$. The correlation charge satisfies $\dot{Q} + [q_1(\vec{v}_1 \times \vec{B}) \cdot \vec{\nabla}_{p_1} + q_2(\vec{v}_2 \times \vec{B}) \cdot \vec{\nabla}_{p_2}]Q = 0$ and thus is zero for all times if it is initially zero. In examining the approach of g to thermodynamic equilibrium, it is even more nonuniform than in the first case because of the oscillations that appear in the argument of ϕ . The Bogolubov approximation, Eq. (3.30), becomes meaningless when we consider the magnetic case with $v_{\parallel} = 0$. The argument of ϕ_{12} will be $\vec{x}_{12} - \vec{v}_{12\perp} \sin B\lambda - \Omega \cdot \vec{v}_{12\perp} \cos B\lambda$. This quantity has no limit as λ tends to infinity. A proper treatment of this problem has recently been obtained by A. Klimas and G. Sandri, Ap. J., 169, 41 (1971). The presence of the magnetic field makes it clear that the relationship between Bogolubov's formula (3.29) and the canonical equilibrium is rather superficial.

Example II - Models of Plasma Instability

We can write the first equation of the BBGKY hierarchy as

$$\frac{\partial F_1}{\partial t} + K_1 F_1 = L_{12} g_{12} + \frac{\partial F_1}{\partial v_1} \cdot \frac{\partial \psi_1}{\partial x_1} \quad (3.33)$$

where we have introduced

$$K_1 = v_1 \cdot \nabla_1, \quad L_{12} = \int dx_2 dv_2 \frac{\partial \phi_{12}}{\partial x_1} \cdot \frac{\partial}{\partial v_1} \quad (3.34)$$

In addition, the Vlasov self-consistent potential has been defined through

$$\psi_1 = \int dx_2 dv_2 \phi_{12} F_2 \quad (3.35)$$

It is easy to show that if the interparticle potential ϕ_{12} is the Coulomb potential, then ψ_1 satisfies the Poisson equation, with the charge generated by the average over the one-body velocity distribution function. The two-particle correlation function, again in full generality, satisfies the equation

$$\begin{aligned} \frac{\partial g_{12}}{\partial t} + K_{12}g_{12} = & \epsilon I_{12}g_{12} + \epsilon I_{12}F_1F_2 + \\ & + \frac{\partial F_1}{\partial v_1} \cdot \frac{\partial \chi_{12}}{\partial x_1} + \frac{\partial F_2}{\partial v_2} \cdot \frac{\partial \chi_{21}}{\partial x_2} + \\ & + \frac{\partial g_{12}}{\partial v_1} \cdot \frac{\partial \psi_1}{\partial x_1} + \frac{\partial g_{12}}{\partial v_2} \cdot \frac{\partial \psi_2}{\partial x_2} \\ & + L^2 h_{123} \end{aligned} \quad (3.36)$$

where we have introduced the correlation potentials given by

$$\chi_{12} = \int dx_3 dv_3 \phi_{13} g_{23} \quad (3.37)$$

$$\chi_{21} = \int dx_3 dv_3 \phi_{23} g_{13} \quad (3.38)$$

The correlation potentials are the terms responsible for Debye shielding and are critical for the understanding of plasma instability. In order to develop a more detailed nonadiabatic analysis for these basic equations, we introduce Fourier transforms and obtain, for a spatially homogeneous plasma, a simpler set of equations which are cast in a form suitable for nonadiabatic analysis; namely,

$$\frac{\partial g_{12}}{\partial t} + \left[D(\vec{k}, \vec{v}_1) + D(-\vec{k}, \vec{v}_2) \right] g_{12} = \tilde{\phi}_1 \vec{k} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F_1 F_2 \quad (3.39)$$

where the linear operator D is defined by

$$D(\vec{k}, \vec{v}) = i\vec{k} \cdot \vec{v} - i\vec{\phi} \vec{k} \cdot \frac{\partial F}{\partial \vec{v}} \int d\vec{v}' \quad (3.40)$$

The explicit form of the operator D can be written as

$$D(\vec{k}, \vec{v}_1)g_{12} = i\vec{k} \cdot \vec{v}_1 g_{12} - i\vec{\phi} \vec{k} \cdot \frac{\partial F_1}{\partial \vec{v}_1} \int d\vec{v}_2 g_{12} \quad (3.41)$$

In the case of the stable plasma, the operator D can be replaced by the first term. Under these conditions, a full nonadiabatic analysis has been carried out showing the causal behavior of g and the need for proper treatment in the neighborhood of small $k \cdot v$.

We can set up a model that allows us to investigate the mathematical behavior of an initially unstable velocity distribution. We write the first two equations of the BBGKY hierarchy as

$$\frac{\partial f}{\partial t} = -\epsilon \int d\vec{x} \, I g = -\epsilon \int d\vec{k} \, \tilde{I} \tilde{g} \quad (3.42)$$

$$\frac{\partial \tilde{g}}{\partial t} + i\vec{k} \cdot \vec{v} \Delta(\vec{k}, \vec{v}) \tilde{g} = \tilde{J} f \quad (3.43)$$

where the tilde represents Fourier transformed quantities. For the time being, the model equations neglect recoil, but they incorporate a most important feature that has to be investigated, namely, the fact that the propagation characteristics of the correlation function are modified by a dielectric response function, Δ , relative to the stable case. In fact, we have

$$\Delta = 1 \quad (\text{stable plasma}) \quad (3.44)$$

and, in general, for a stable plasma,

$$\Delta^*(\vec{k}) = \Delta(-\vec{k}) \quad (3.45)$$

In the stable case, we have a zero for the D function

$$D(k, v) = i k \cdot v \Delta(k, v) \quad (3.46)$$

at

$$\mathbf{k} \cdot \mathbf{v} = 0 \quad (3.47)$$

This condition corresponds to

- (a) $\mathbf{k} = 0$ (long wavelengths)
- (b) $\mathbf{v} = 0$ (zero relative velocity)
- (c) \mathbf{k} orthogonal to \mathbf{v} (mode propagating perpendicular to the particle velocity)

A zero for the D function that is fundamentally different from the stable zero is given by

$$D = |\mathbf{k}| \quad (3.48)$$

The behavior of f for this case has been obtained by a rather difficult numerical integration. The method employed consists of transforming the coupled equations for f and g into an integro-differential equation for f alone. This latter is of the Fahlung form for Laplace transforms and therefore its solution can be obtained in the Laplace variable. The inversion of the Laplace transform was then obtained with the Bellman-Kalaba-Lockett inversion method. The result of the numerical integration is shown in Figure 1. Extensive numerical analysis shows that the qualitative features of the result are insensitive to the specific value chosen for ϵ .

To construct the integrodifferential equation for f , we solve the \tilde{g} equation as

$$\tilde{g} = \int_0^t d\lambda e^{-i\Delta \mathbf{k} \cdot \mathbf{v}(t-\lambda)} \tilde{J}(\mathbf{k}) f(\lambda) \quad (3.49)$$

Therefore,

$$\frac{\partial f}{\partial t} = -\epsilon \int_0^t d\lambda k(t-\lambda) f(\lambda) \quad (3.50)$$

where the kernel function is given by

$$k(t) = \int_{-\infty}^{+\infty} d\mathbf{k} \tilde{I}(\mathbf{k}) \tilde{J}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{v} \Delta t} \quad (3.51)$$

and the k integration is in the direction parallel to the relative velocity. With

$$\tilde{I} = \tilde{J} = \frac{1}{\sqrt{2}} e^{-|k|/2} \quad (3.52)$$

$$\Delta = \frac{|k|}{ik \cdot v} \quad (3.53)$$

we readily obtain

$$k(t) = \frac{1}{1+t} \quad (3.54)$$

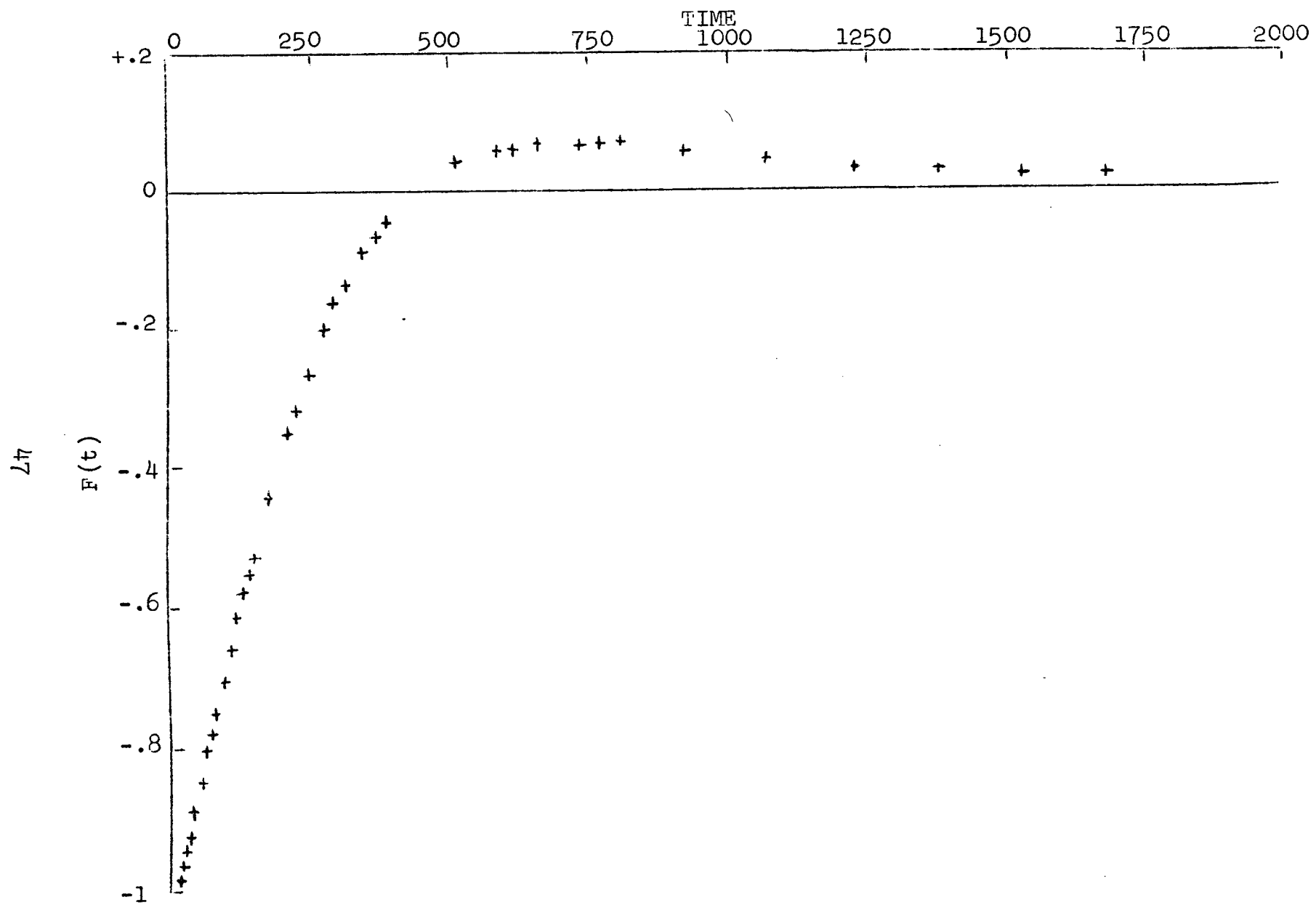


Figure 1

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